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Some Integrals of the Dedekind η-Function

M.L. Glasser
Department of Physics
Clarkson University, Potsdam, NY 13699-5280(USA)

e-mail: laryg@clarkson.edu

Abstract

This note presents selected values of the class of integrals

$$\int_0^\infty f(x)\eta^n(ix)dx.$$

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The motivation for this study was to investigate integrals similar to one that Ramanujan included as item IV(5) in his famous 1913 letter to Hardy [1] containing the expression $\cosh(x) + \cos(x)$ in the denominator. Actually, the first to examine an integral of this class appears to have been J.W.L. Glaisher[2] in 1871; it seems doubtful, however, that Ramanujan would have been aware of this work. The study of these integrals quickly led to the problem of evaluating integrals of the Dedekind Eta function, and particularly pertaining to the class

$$\int_0^\infty f(x)\eta^n(ix)dx \tag{1}$$

This is probably to have been expected, since Ramanujan devotes a good deal of space in his notebooks to such integrals. These and many others can be found in Berndt's seminal exposition of Ramanujan's work[3]. Other Dedekind function integrals appear scattered in the literature, such as in references [4,5] and works cited there. The aim of this paper is to present a number of results of the form (1) having a more elementary character.

Note that for the real nome $q = e^{-2\pi x}$, our instance of the Dedekind η-function is given by

$$\eta(ix) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{2}$$

We begin with a simple rearrangement of Euler's identity[6]

$$\eta(ix) = \frac{2}{\sqrt{3}}\sum_{n=0}^{\infty}\cos[(2n+1)\pi/6]q^{(2n+1)^2/24}. \tag{3}$$

Then, after a change of variable $q = e^{-2\pi x}$, one has

$$\begin{aligned}\int_0^1 \frac{dq}{q}q^y\eta(ix) &= \frac{2\pi}{\sqrt{3}}\sum_{n=0}^{\infty}\cos[(2n+1)\frac{\pi}{6}]\int_0^\infty dx\; e^{-2\pi[y+(2n+1)^2/24]x} \\ &= \frac{48\pi}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{\cos[(2n+1)\pi/6]}{(2n+1)^2+24y} = \pi\sqrt{\frac{2}{y}}\frac{\sinh\pi\sqrt{8y/3}}{\cosh\pi\sqrt{6y}}.\end{aligned} \tag{4}$$

We therefore have the Laplace transform, where $t = 3\pi y$,

$$\int_0^\infty e^{-xt}\eta(ix)dx = \sqrt{\frac{\pi}{t}}\frac{\sinh 2\sqrt{\pi t/3}}{\cosh\sqrt{3\pi t}}. \tag{5}$$

In what follows, we shall assume that all parameters take on real values for which the integrals converge absolutely.

Let $F(t)$ denote the inverse Laplace transform of $f(x)$. Then from (5) by multiplying both sides by $F(t)$ and integrating over t,we have, by invoking

Fubini's theorem, the identity

$$\int_0^\infty f(x)\eta(ix)dx = \sqrt{\pi}\int_0^\infty \frac{dt}{\sqrt{t}}F(t)\frac{\sinh 2\sqrt{\pi t/3}}{\cosh\sqrt{3\pi t}}. \tag{6}$$

We obtain in this way[8]

$$\begin{aligned}\int_0^\infty x^{-s}\eta(ix)dx = \\ \frac{8\sqrt{3}\pi}{16^s(3\pi)^s}\frac{\Gamma(2s-1)}{\Gamma(s)}[\zeta(2s-1,\frac{1}{12})+\zeta(2s-1,\frac{11}{12}) \\ -\zeta(2s-1,\frac{5}{12})-\zeta(2s-1,\frac{7}{12})], \quad (s>0).\end{aligned} \tag{7}$$

Next, in (5) replace t by it and formally take the real part of both sides to get

$$\int_0^\infty \cos(xy)\eta(ix)dx = \sqrt{\frac{\pi}{2y}}\frac{\sinh\sqrt{8\pi y/3}+\sin\sqrt{8\pi y/3}}{\cosh\sqrt{8\pi y/3}+\cos\sqrt{8\pi y/3}}. \tag{8}$$

For $y \to 0$, (8) becomes

$$\int_0^\infty \eta(ix)dx = \frac{2\pi}{\sqrt{3}}. \tag{9}$$

Similarly,

$$\int_0^\infty \sin(xy)\eta(ix)dx = \sqrt{\frac{\pi}{2y}}\frac{\sinh\sqrt{8\pi y/3}-\sin\sqrt{8\pi y/3}}{\cosh\sqrt{8\pi y/3}+\cos\sqrt{8\pi y/3}}. \tag{10}$$

By dividing both sides of (10) by y and integrating over y from 0 to ∞, we obtain

$$\int_0^\infty \frac{dx}{x^2}\frac{\sinh(x)-\sin(x)}{\cosh(x)+\cos(x)} = \frac{\pi}{4} \tag{11}$$

an integral of the type that led to this study.

Next, Jacobi's triple identity [7] may be written

$$\eta^3(ix) = \sum_{n=0}^\infty (-1)^n(2n+1)q^{(2n+1)^2/8}. \tag{12}$$

By multiplying both sides by q^{z-1} and integrating over $0<q<1$ as before, we find, with $q=e^{-2\pi x}$,

$$\int_0^1 \frac{dq}{q}q^z\eta^3(ix) = 8\sum_{n=0}^\infty \frac{(-1)^n(2n+1)}{(2n+1)^2+8z} = \frac{2\pi}{\cosh(\pi\sqrt{2z})}, \tag{13}$$

which gives us the Laplace transform

$$\int_0^\infty e^{-xy}\eta^3(ix)dx = \mathrm{sech}\sqrt{\pi \mathrm{y}}. \tag{14}$$

As for (6), if $F(t)$ denotes the inverse Laplace transform of $f(x)$, we have the identity

$$\int_0^\infty f(x)\eta^3(ix)dx = \int_0^\infty \frac{F(t)}{\cosh\sqrt{\pi \mathrm{t}}}dt. \tag{15}$$

Formula (15) is much more flexible than (6) and leads to a variety of interesting looking integrals, a number of which are displayed in the appendix, including the mysterious (A7)

$$\int_0^\infty \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}}e^{-\pi x/4}\prod_{n=0}^\infty (1-e^{-2\pi nx})^3 dx = \sqrt{2}-1. \tag{16}$$

This is derived by noting that the Laplace transform of $F(t) = \sin(at)/\sqrt{\pi t}$ is the Imaginary part of $(a+ix)^{-1/2}$. Inserting this into (15) and using Parseval's identity for the cosine Fourier transform to simplify the integral on the right hand side, one obtains (16) for $a = 1$

By proceeding analogously to the derivation of (11), we find a second example of the "Glaisher-Ramanujan" class

$$\int_0^\infty \frac{dx}{x}\frac{\sinh(x/2)\sin(x/2)}{\cosh(x)+\cos(x)} = \frac{\pi}{8}. \tag{17}$$

In conclusion, we have opened a way to produce many integrals over the Dedekind-η function and evaluated one or two Glaisher-Ramnujan integrals. But, there exist many further $q-$ identities similar to (3) and (12) which might prove productive for extending this investigation.

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Appendix

$$\int_0^\infty e^{-xy}\eta^3(ix)\,dx = \mathrm{sech}\sqrt{\pi y} \tag{A1}$$

$$\int_0^\infty \frac{\eta^3(ix)}{x+a}\,dx = \frac{2}{\pi}\int_0^\infty \frac{xe^{-ax^2/\pi}}{\cosh(x)}\,dx \tag{A2}$$

$$\int_0^\infty x^{-\nu}\eta^3(ix)\,dx = \frac{4}{\pi^\nu}\frac{\Gamma(2\nu)}{\Gamma(\nu)}\beta(2\nu) \qquad (\nu > 0) \tag{A3}$$

$$\int_0^\infty \frac{\eta^3(ix)}{\sqrt{x+a}}\,dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-ax^2/\pi}}{\cosh(x)}\,dx \tag{A4}$$

$$\int_0^\infty x^{-1/2}e^{-a/x}\eta^3(ix)\,dx = \mathrm{sech}\sqrt{\pi a} \tag{A5}$$

$$\int_0^\infty e^{-xy}\eta^3(ix)\frac{dx}{x} = \frac{2}{\pi}\int_{\sqrt{\pi y}}^\infty x\,\mathrm{sech}(x)\,dx \tag{A6}$$

$$\int_0^\infty \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}}\,\eta^3(ix)\,dx = \sqrt{2}-1 \tag{A7}$$

$$\int_0^\infty x^{-1/2}\cos(a/x)\eta^3(ix)\,dx = 2\,\frac{\cos\sqrt{\pi a/2}\cosh\sqrt{\pi a/2}}{\cos\sqrt{2\pi a}+\cosh\sqrt{2\pi a}} \tag{A8}$$

$$\int_0^\infty x^{-1/2}\mathrm{erf}(\sqrt{bx})\eta^3(ix)\,dx = \frac{4}{\pi}\arctan(\tanh\frac{1}{2}\sqrt{\pi b}) \tag{A9}$$

$$\int_0^\infty x^{-1/2}e^{a/x}\mathrm{erfc}(\sqrt{a/x})\eta^3(ix)\,dx =$$

$$\frac{1}{\pi\sqrt{a}}\left[\psi(\frac{1}{2}\sqrt{a/\pi}+\frac{3}{4})-\psi(\frac{1}{2}\sqrt{a/\pi}+\frac{1}{4})\right] \tag{A10}$$

$$\int_0^\infty \cos(xy)\eta^3(ix)dx = \frac{\cosh(\sqrt{\pi y/2})\cos(\sqrt{\pi y/2})}{\sinh^2(\sqrt{\pi y/2})+\cos^2(\sqrt{\pi y/2})} \tag{A11}$$

$$\int_0^\infty \sin(xy)\eta^3(ix)dx = \frac{\sinh(\sqrt{\pi y/2})\sin(\sqrt{\pi y/2})}{\sinh^2(\sqrt{\pi y/2})+\cos^2(\sqrt{\pi y/2})} \tag{A12}$$

$$\int_0^\infty \eta(ix)dx = \frac{2\pi}{\sqrt{3}} \tag{A13}$$

$$\int_0^\infty \eta^3(ix)dx = 1 \tag{A14}$$

$$\int_0^\infty x^n\eta^3(ix)dx = \frac{4n!}{\pi^{n+1}}\beta(2n+1) \tag{A15}$$

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